

REMARKS ON COUNTABLE TIGHTNESS

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ABSTRACT. Countable tightness may be destroyed by countably closed forcing. We characterize the indestructibility of countable tightness under countably closed forcing by combinatorial statements similar to the ones Tall used to characterize indestructibility of the Lindelöf property under countably closed forcing. We consider the behavior of countable tightness in generic extensions obtained by adding Cohen reals. We show that HFD's are indestructibly countably tight.

Let (X, τ) be a topological space and let x be an element of X . We say that X is *countably tight at x* if there is for each set $A \subseteq X$ with $x \in \overline{A}$, a countable set $B \subseteq A$ such that $x \in \overline{B}$. If the space is countably tight at each of its elements, we say that the space has countable tightness or, equivalently, that the space is countably tight.

It is known that in generic extensions by countably closed posets a groundmodel space that is countably tight may fail, in the generic extension, to still be a countably tight space. In [5] Dow gives a ingenious proof, using reflection arguments, that in the generic extension by an iteration of first the Cohen poset for adding uncountably many Cohen reals, then any countably closed poset, countably tight topological spaces from the ground model remain countably tight. A similar phenomenon regarding the preservation of the Lindelöf property has been shown by Dow in [4]. In [13] we gave an explanation for this phenomenon for Lindelöf spaces. We now show that for reasons very analogous to the Lindelöf case, this preservation happens for countable tightness.

1. INDESTRUCTIBILITY OF COUNTABLE TIGHTNESS BY COUNTABLY CLOSED FORCING

By analogy with the Lindelöf case in [15], we say that a topological space is *indestructibly countably tight* if the space is countably tight, and in any generic extension by countably closed forcing the space is still countably tight. For convenience define, for $x \in X$ not an isolated point of X , $\Omega_x = \{A \subseteq X : x \in \overline{A} \setminus A\}$. Thus, X is countably tight at x if each element of Ω_x has a countable subset which is in Ω_x . From now on, assume that x is an element of X , and that X is countably tight at x . Following [15] for the corresponding notion for Lindelöf spaces, we define:

Definition 1. A set $T = \{y_f : f \in \cup_{\alpha < \omega_1} {}^\alpha\omega\} \subseteq X$ is an x -tightness tree if for each $\alpha < \omega_1$ and for each $f \in {}^\alpha\omega$ we have $\{y_{f \cup \{(\alpha, n)\}} : n < \omega\} \in \Omega_x$.

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We also introduce the following infinite two-person game of length α , denoted $G_1^\alpha(\Omega_x, \Omega_x)$: In inning $\beta < \alpha$ player ONE first selects an $O_\beta \in \Omega_x$, and TWO responds with an $x_\beta \in O_\beta$. A play $O_0, x_0, O_1, x_1, \dots, O_\beta, x_\beta, \dots$ $\beta < \alpha$ is won by Player TWO if $\{x_\beta : \beta < \alpha\} \in \Omega_x$; else, ONE wins. In [11] this game was examined for the case when $\alpha = \omega$.

Theorem 1. *For a topological space X which is countably tight at the element $x \in X$, the following are equivalent:*

- (1) X is indestructibly countably tight at x .
- (2) The countable tightness of X at x is preserved upon forcing with $\text{Fn}(\omega_1, \omega, \omega_1)$, the poset for adding a Cohen subset of ω_1 with countable conditions.
- (3) For each x -tightness tree $\{y_f : f \in \cup_{\alpha < \omega_1} {}^\alpha \omega\}$ the set $\{g \in \cup_{\alpha < \omega_1} {}^\alpha \omega : \{y_{g \upharpoonright \gamma} : \gamma < \text{dom}(g)\} \in \Omega_x\}$ is dense in $\text{Fn}(\omega_1, \omega, \omega_1)$.
- (4) For each x -tightness tree $\{y_f : f \in \cup_{\alpha < \omega_1} {}^\alpha \omega\}$ and for each $g \in \cup_{\alpha < \omega_1} {}^\alpha \omega$ there is an $f \in {}^{\omega_1} \omega$ such that $f \upharpoonright_{\text{dom}(g)} = g$ and $\{y_{f \upharpoonright \alpha} : \alpha < \omega_1\} \in \Omega_x$.
- (5) For each x -tightness tree $\{y_f : f \in \cup_{\alpha < \omega_1} {}^\alpha \omega\}$ there is an $f \in \cup_{\alpha < \omega_1} {}^\alpha \omega$ such that $\{y_{f \upharpoonright \alpha} : \alpha < \text{dom}(f)\} \in \Omega_x$.
- (6) ONE does not have a winning strategy in the game $G_1^{\omega_1}(\Omega_x, \Omega_x)$.

Proof. That (2) \Rightarrow (1): We follow the argument in [14], adapted to the current context. Thus, let $(\mathbb{P}, <)$ be a countably closed partially ordered set, and assume that $\mathbf{1}_{\mathbb{P}} \Vdash \text{“}\dot{X} \text{ is not countably tight at } \dot{x}\text{.”}$ Fix a \mathbb{P} -name \dot{A} and a $p \in \mathbb{P}$ such that $p \Vdash \text{“}\dot{A} \in \Omega_x \text{ but for each countable } C \subseteq \dot{A}, C \notin \Omega_x\text{.”}$ Put $p_\emptyset = p$ and let η_\emptyset be the empty sequence of length 0. Define $F_\emptyset = \{y \in X : (\exists q \leq p_\emptyset)(q \Vdash \text{“}\dot{y} \in \dot{A}\text{”})\}$.

Claim 1: $F_\emptyset \in \Omega_x$:

Suppose that on the contrary x is not in the closure of F_\emptyset . Choose a neighborhood U of x with $U \cap F_\emptyset = \emptyset$. As $p \Vdash \text{“}\dot{A} \in \Omega_x\text{”}$ we find that $p \Vdash \text{“}\dot{U} \cap \dot{A} \neq \emptyset\text{”}$. Choose a $q \leq p$ and a $y \in X$ such that $q \Vdash \text{“}\dot{y} \in \dot{U} \cap \dot{A}\text{”}$. Then y is in F_\emptyset , and as y is in U we find the contradiction that $U \cap F_\emptyset \neq \emptyset$. This completes the proof of Claim 1.

As X is countably tight, choose a countable $C_\emptyset \subseteq F_\emptyset$ with $x \in \overline{C_\emptyset}$. Enumerate C_\emptyset bijectively as $(y_n : n < \omega)$. For each n choose $p_{(n)} < p_\emptyset$ such that $p_{(n)} \Vdash \text{“}\dot{y}_n \in \dot{A}\text{”}$.

This specifies p_η and y_η for $\eta \in {}^1 \omega$. Now let $0 < \alpha < \omega_1$ be given, and assume that for each $\beta < \alpha$ and $\sigma \in {}^\beta \omega$ we have selected $p_\sigma \in \mathbb{P}$ and $y_\sigma \in X$ such that for all $\gamma < \beta$ we have $\{y_{\sigma \upharpoonright \gamma \smallfrown \{(\gamma, n)\}} : n < \omega\} \in \Omega_x$ and $p_\sigma \Vdash \text{“}\dot{y}_\sigma \in \dot{A}\text{”}$.

Now we distinguish two cases: α is a limit ordinal, or α is a successor ordinal.

Case 1: α is a limit ordinal. Then for each $\sigma \in {}^\alpha \omega$ choose a $p_\sigma \in \mathbb{P}$ such that for each $\beta < \alpha$ we have $p_\sigma < p_{\sigma \upharpoonright \beta}$. This is possible since \mathbb{P} is countably closed.

Case 2: α is a successor ordinal. Say $\alpha = \beta + 1$. For each $\sigma \in {}^\beta \omega$ define

$$F_\sigma = \{y \in X : (\exists q \leq p_\sigma)(q \Vdash \text{“}\dot{y} \in \dot{A}\text{”})\}.$$

Claim 2: $F_\sigma \in \Omega_x$:

The proof of Claim 2 proceeds like the proof of Claim 1.

As X is countably tight, choose a countable $C_\sigma \subseteq F_\sigma$ with $x \in \overline{C_\sigma}$. Enumerate C_σ bijectively as $(y_{\sigma \cup \{(\beta, n)\}} : n < \omega)$ and for each n choose $p_{\sigma \cup \{(\beta, n)\}} < p_\sigma$ such that $p_{\sigma \cup \{(\beta, n)\}} \Vdash \text{“}\dot{y}_{\sigma \cup \{(\beta, n)\}} \in \dot{A}\text{”}$.

This defines p_τ for each τ in ${}^\alpha \omega$, and when α is a successor ordinal this also defines each y_τ .

We now show that in the generic extension by $\text{Fn}(\omega_1, \omega, \omega_1)$ X fails to have countable tightness. For let $g \in {}^{\omega_1}\omega$ be $\text{Fn}(\omega_1, \omega, \omega_1)$ -generic, and put

$$B = \{y_{g \upharpoonright \alpha} : \alpha < \omega_1\}.$$

Claim 3: $B \in \Omega_x$.

For consider any open neighborhood U of x . Then $D_U = \{\eta \in \cup_{\alpha < \omega_1} {}^\alpha\omega : y_\eta \in U\}$ is a dense subset of $\text{Fn}(\omega_1, \omega, \omega_1)$. For let any $p \in \text{Fn}(\omega_1, \omega, \omega_1)$ be given. We may assume that $\text{dom}(p) = \alpha < \omega_1$. Since $\{y_{p \upharpoonright \{(\alpha, n)\}} : n < \omega\}$ is a member of Ω_x , we have $U \cap \{y_{p \upharpoonright \{(\alpha, n)\}} : n < \omega\} \neq \emptyset$. Choose $n < \omega$ with $y_{p \upharpoonright \{(\alpha, n)\}} \in U$. Then $q = p \upharpoonright \{(\alpha, n)\} \in D_U$ and $q < p$. Since the generic filter producing g meets this dense set we find that $U \cap B \neq \emptyset$. It follows that B is a member of Ω_x .

Claim 4: No countable subset of B is in Ω_x .

For fix a $\beta < \omega_1$ and consider $B_\beta = \{y_{g \upharpoonright \gamma} : \gamma < \beta\}$. If it were the case that B_β is an element of Ω_x , then we would have $p_\beta \Vdash \text{“}\check{B}_\beta \subseteq \dot{A}\text{”}$ and $p_\beta \Vdash \text{“}\check{B}_\beta \in \Omega_x\text{”}$. This contradicts the selection of the \mathbb{P} -name \dot{A} .

That (1) \Rightarrow (3): Assume that (3) fails. Fix an x -tightness tree $\{y_f : f \in \cup_{\alpha < \omega_1} {}^\alpha\omega\}$ that witnesses this failure. Since the set $D := \{g \in \cup_{\alpha < \omega_1} {}^\alpha\omega : \{y_{g \upharpoonright \gamma} : \gamma < \text{dom}(g)\} \in \Omega_x\}$ is not dense in $\text{Fn}(\omega_1, \omega, \omega_1)$, fix a $p \in \text{Fn}(\omega_1, \omega, \omega_1)$ for which there is no $g \in D$ with $g < p$. We may assume that $\text{dom}(p) = \alpha < \omega_1$. Then for each $g \in \cup_{\alpha < \omega_1} {}^\alpha\omega$ with $g < p$ we have $\{y_{g \upharpoonright \gamma} : \gamma < \text{dom}(g)\}$ is not in Ω_x . But then for each generic filter G of $\text{Fn}(\omega_1, \omega, \omega_1)$ which contains p , if h is the corresponding generic element, then $B = \{y_{h \upharpoonright \alpha} : \alpha < \omega_1\}$ is a member of Ω_x , but no countable subset of it is in Ω_x . But then $\mathbb{P} = \{g \in \text{Fn}(\omega_1, \omega, \omega_1) : g < p\}$ with the inherited order of $\text{Fn}(\omega_1, \omega, \omega_1)$ is a countably closed partially ordered set forcing that X is not countably tight.

That (3) \Rightarrow (4) and (4) \Rightarrow (5): These implications follow directly.

That (5) \Rightarrow (6): A strategy of F ONE together with the fact that X is a space of countable tightness provides an x -tightness tree as follows:

F calls on ONE to play members of Ω_x . As X is of countable tightness, we may assume that ONE's moves are countable elements of Ω_x . Thus, enumerate $F(\emptyset)$ bijectively as $\{y_{\{(0, n)\}} : n < \omega\}$. For $\alpha < \omega_1$ assume that we have already defined for each $\beta < \alpha$ and each $g \in {}^\beta\omega$ a $y_g \in X$ such that $\{y_{g \upharpoonright \gamma \cup \{(\gamma, n)\}} : n < \omega\} = F(y_{g \upharpoonright 0}, \dots, y_{g \upharpoonright \gamma}) \in \Omega_x$, ($\gamma < \beta$).

Case 1: $\alpha = \beta + 1$, a successor ordinal.

Then we define $\{y_{g \upharpoonright \{(\beta, n)\}} : n < \omega\} = F(y_{g \upharpoonright 0}, \dots, y_{g \upharpoonright \gamma}, \dots, y_g)$

Case 2: α is a limit ordinal.

In this case we choose $y_g \in F(g \upharpoonright \gamma : \gamma < \alpha)$ arbitrarily.

But then the set $\{y_g : g \in \cup_{\alpha < \omega_1} {}^\alpha\omega\}$ is an x -tightness tree. Applying (5) we fix an $f \in \cup_{\alpha < \omega_1} {}^\alpha\omega$ such that $\{y_{f \upharpoonright \beta} : \beta < \text{dom}(f)\} \in \Omega_x$. But then f codes a play of the game against F in which TWO won. This shows that F is not a winning strategy for ONE in $\mathbb{G}_1^{\omega_1}(\Omega_x, \Omega_x)$.

That (6) \Rightarrow (1):

Let $(\mathbb{P}, <)$ be a countably closed partially ordered set and let X be a topological space that is countably tight at $x \in X$.

Let \dot{A} be a \mathbb{P} name such that $\mathbf{1}_{\mathbb{P}} \Vdash \text{“}\check{x} \in \overline{\dot{A}}\text{”}$. Choose an arbitrary member p of \mathbb{P} . We now use ideas as in the proof of (2) \Rightarrow (1) to define a strategy F of ONE in the game $\mathbb{G}_1^{\omega_1}(\Omega_x, \Omega_x)$.

To begin, define $H_\emptyset = \{y \in X : (\exists q \leq p)(q \Vdash \text{"}\check{y} \in \dot{A}\text{"})\}$. As in Claim 1 above, $H_\emptyset \in \Omega_x$. Since X has countable tightness at x , choose $F(\emptyset) = C_\emptyset \subseteq H_\emptyset$ countable with $x \in \overline{C_\emptyset}$. Enumerate C_\emptyset as $(y_{(n)} : n < \omega)$. For each n choose $p_{(n)} < p$ such that $p_{(n)} \Vdash \text{"}\check{y}_{(n)} \in \dot{A}\text{"}$. Now $H_\emptyset, F(\emptyset), p_{(n)}, n < \omega$ and $y_{(n)} n < \omega$ are specified.

To describe the rest of the recursive construction of ONE's strategy F , suppose that $0 < \alpha < \omega_1$ is given, and that for each $\gamma < \alpha$, and each $\sigma \in {}^\gamma\omega$ we already have specified:

the set H_σ , a countable subset C_σ of H_σ , element p_σ of \mathbb{P} , and if γ is a successor ordinal, $y_\sigma \in X$ such that

- (1) $H_\sigma = \{y \in X : (\exists q \leq p_\sigma)(q \Vdash \text{"}\check{y} \in \dot{A}\text{"})\}$ is in Ω_x ;
- (2) $F(y_\nu : \nu \subset \sigma \text{ and } \text{dom}(\nu) \text{ a successor ordinal}) = C_\sigma \subseteq H_\sigma$ is a countable set which is a member of Ω_x ;
- (3) If $\xi = \text{dom}(\sigma) < \alpha$ then $C_\sigma = \{y_{\sigma \cup \{(\xi, n)\}} : n < \omega\}$.
- (4) $p_\sigma < p_\nu$ for each $\nu \in {}^\xi\omega$ with $\nu \subset \sigma$;
- (5) If $\text{dom}(\sigma)$ is a successor ordinal, then $p_\sigma \Vdash \text{"}\check{y}_\sigma \in \dot{A}\text{"}$;

We must now specify these parameters for $\tau \in {}^\alpha\omega$.

Case 1: $\alpha = \beta + 1$, a successor ordinal. Consider any $\sigma \in {}^\beta\omega$. Since p_σ is already defined, we have $H_\sigma = \{y \in X : (\exists q \leq p_\sigma)(q \Vdash \text{"}\check{y} \in \dot{A}\text{"})\}$ where as in Claim 1 above, H_σ is an element of Ω_x . By the countable tightness of X at x the countable set $C_\sigma \subseteq H_\sigma$ is selected such that C_σ is an element of Ω_x , and we define

$$F(y_\nu : \nu \subset \sigma \text{ and } \text{dom}(\nu) \text{ a successor ordinal}) = C_\sigma.$$

By enumerating C_σ as $\{y_{\sigma \cup \{(\beta, n)\}} : n < \omega\}$ we specify y_τ for each τ in ${}^\alpha\omega$ which extends σ . Then for each of these y_τ we choose a $p_\tau < p_\sigma$ such that $p_\tau \Vdash \text{"}\check{y}_\tau \in \dot{A}\text{"}$.

Case 2: α is a limit ordinal. For $\sigma \in {}^\alpha\omega$ choose, by the countable closedness of \mathbb{P} , a $p_\sigma \in \mathbb{P}$ such that for each initial segment ν of σ we have $p_\sigma < p_\nu$ and then define

$$H_\sigma = \{y \in X : (\exists q \leq p_\sigma)(q \Vdash \text{"}\check{y} \in \dot{A}\text{"})\}.$$

As before H_σ is a member of Ω_x , and the recursive construction can continue.

Since the defined F is a strategy for ONE in the game $G_1^{\omega_1}(\Omega_x, \Omega_x)$, our hypothesis implies that F is not a winning strategy for ONE. Thus, choose an F -play lost by ONE. This play is of the following form: For an $f \in {}^{\omega_1}\omega$ we have the sequence

$$((F(y_{f \restriction \beta} : \beta < \alpha \text{ a successor ordinal}), y_{f \restriction \alpha}) : \alpha < \omega_1 \text{ a successor ordinal})$$

for which the set $\{y_{f \restriction \alpha} : \alpha < \omega_1 \text{ a successor ordinal}\}$ of player TWO's moves in the play is a member of Ω_x . Using the countable tightness of X at x again, we find that there is a countable ordinal $\beta < \omega_1$ for which the set

$$D = \{y_{f \restriction \alpha} : \alpha < \beta \text{ a successor ordinal}\}$$

is in Ω_x . But then we have $p_{f \restriction \beta} \Vdash \text{"}\check{D} \subseteq \dot{A} \text{ is a countable element of } \Omega_{\check{x}}\text{"}$.

Thus, we find that for each p in \mathbb{P} there is a $q < p$ which forces that x is in the closure of some countable subset of \dot{A} . It follows that

$$\mathbf{1}_{\mathbb{P}} \Vdash \text{"}\check{X} \text{ has countable tightness at } \check{x}\text{"} \quad \square$$

2. COUNTABLE STRONG FAN TIGHTNESS AND COHEN REALS

Recall that for families \mathcal{A} and \mathcal{B} of sets the symbol $S_1(\mathcal{A}, \mathcal{B})$ denotes the statement that there is for each sequence $(O_n : n \in \mathbb{N})$ of elements of \mathcal{A} a corresponding sequence $(x_n : n \in \mathbb{N})$ such that for each n we have $x_n \in O_n$, and $\{x_n : n \in \mathbb{N}\} \in \mathcal{B}$.

Sakai defined the notion of countable strong fan tightness at x , which in our notation is $S_1(\Omega_x, \Omega_x)$, in [10]. It is clear that if for some countable ordinal α ONE has no winning strategy in the game $G_1^\alpha(\Omega_x, \Omega_x)$, then the space has countable strong fan tightness at x . It is not in general true that if a space has countable strong fan tightness at a point x , then ONE has no winning strategy in the game $G_1^\omega(\Omega_x, \Omega_x)$ - see pp. 250 - 251 of [11] for an ad hoc example. In Theorem 8 below we give another proof of this fact under the Continuum Hypothesis, CH.

From Theorem 1 it follows that spaces where ONE does not have a winning strategy in the game $G_1^\omega(\Omega_x, \Omega_x)$, are indestructibly countably tight at x . We now show, by rewriting the proof of (1) \Rightarrow (2) of Theorem 13B of [11] into the forcing context, that a space countably tight at a point x is converted to a space in which ONE has no winning strategy in the game $G_1^\omega(\Omega_x, \Omega_x)$ in the generic extension by uncountably many Cohen reals. For uncountable cardinals κ let $\mathbb{C}(\kappa)$ denote the poset for adding κ Cohen reals.

We use the following lemma of Dow [5], Lemma 5.2:

Lemma 2 (Dow). *Let κ be an infinite cardinal and let X be a topological space which is countably tight at $x \in X$. Then*

$$\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{“}\check{X} \text{ is countably tight at } \check{x}\text{.”}$$

Theorem 3. *Let κ be an uncountable cardinal. If (X, τ) is a topological space of countable tightness at x , then*

$$\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{“ONE has no winning strategy in the game } G_1^\omega(\Omega_x, \Omega_x)\text{”}.$$

Proof. Let $\dot{\sigma}$ be a $\mathbb{C}(\kappa)$ name such that

$$\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{“}\dot{\sigma} \text{ is a strategy of ONE in } G_1^\omega(\Omega_x, \Omega_x)\text{.”}$$

By Lemma 2 $\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{“}\check{X} \text{ is countably tight at } \check{x}\text{.”}$ Therefore we have

$$\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{“}\dot{\sigma}(\emptyset) \text{ has a countable subset which is a member of } \Omega_{\check{x}}\text{.”}$$

Choose a $\mathbb{C}(\kappa)$ name \dot{O}_\emptyset such that

$$\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{“}\dot{O}_\emptyset \subseteq \dot{\sigma}(\emptyset) \text{ is a countable subset with } \check{x} \text{ in its closure.”}$$

Thus choose $\mathbb{C}(\kappa)$ names \dot{y}_n , $n < \omega$ such that $\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{“}\dot{C}_\emptyset = \{\dot{y}_n : n < \omega\}\text{.”}$ Then we have $\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{“}(\forall n)(\dot{\sigma}(\dot{y}_n) \text{ has a countable subset which is in } \Omega_{\check{x}})\text{.”}$ For each n we choose $\mathbb{C}(\kappa)$ names \dot{C}_n and $\dot{y}_{n,k}$, $k < \omega$ such that

$$\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{“}\dot{C}_n \subseteq \dot{\sigma}(\dot{y}_n) \text{ is a countable subset with } \check{x} \text{ in its closure”}$$

and $\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{“}\dot{C}_n = \{\dot{y}_{n,k} : k < \omega\}\text{”}$ and so on. In this way we find for each finite sequence n_1, \dots, n_k of elements of ω $\mathbb{C}(\kappa)$ names $\dot{C}_{n_1, \dots, n_k}$ and $\dot{y}_{n_1, \dots, n_k}$ such that

$$\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{“}\{\dot{y}_{n_1, \dots, n_k, m} : m < \omega\} = \dot{C}_{n_1, \dots, n_k}\text{”}$$

and

$$\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{“}\dot{C}_{n_1, \dots, n_k} \subseteq \dot{\sigma}(\dot{y}_{n_1, \dots, n_k})\text{”}$$

and

$$\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \dot{C}_{n_1, \dots, n_k} \text{ is a countable set in } \Omega_{\check{x}}$$

Since $\mathbb{C}(\kappa)$ has countable chain condition and each of the names \dot{y}_τ and \dot{C}_τ is a name for a single element of X or a countable set of elements of X , there is an $\alpha < \kappa$ such that each of these is a $\mathbb{C}(\alpha)$ name. Thus, factoring the forcing as $\mathbb{C}(\alpha) * \mathbb{C}([\alpha, \kappa])$ we may assume that all the named objects are in the ground model. Then, in the generic extension by $\mathbb{C}([\alpha, \kappa])$ over this ground model there is a function $f \in {}^\omega \omega$ such that f is not in any first category set definable from parameters in the ground model.

Now for each neighborhood V of x in the ground model define, in the ground model $F_V = \{f \in {}^\omega \omega : (\forall k)(y_{f \upharpoonright k} \notin V)\}$ is first category and is definable from parameters in the ground model only. Thus, in the generic extension by $\mathbb{C}([\alpha, \kappa]) \cup \{F_V : V \text{ a neighborhood of } x\} \neq {}^\omega \omega$. Choose in this generic extension an f with

$$f \in {}^\omega \omega \setminus \bigcup \{F_V : V \text{ a ground model neighborhood of } x\}$$

Then in the generic extension the σ -play during which TWO selected the sets $y_{f \upharpoonright n}$, $0 < n < \omega$ is won by TWO. This completes the proof that in the generic extension ONE has no winning strategy in the game $G_1(\Omega_x, \Omega_x)$ on X . \square

And the property that ONE has no winning strategy in the game $G_1^\omega(\Omega_x, \Omega_x)$ is preserved by countably closed forcing:

Theorem 4. *If (X, τ) is a topological space for which ONE has no winning strategy in the game $G_1^\omega(\Omega_x, \Omega_x)$, then for any countably closed partially ordered set \mathbb{P} ,*

$$\mathbf{1}_{\mathbb{P}} \Vdash \text{“ONE has no winning strategy in the game } G_1^\omega(\Omega_x, \Omega_x)\text{”}.$$

Proof. Let (X, τ) be a topological space for which ONE has no winning strategy in the game $G_1^\omega(\Omega_x, \Omega_x)$. Let $(\mathbb{P}, <)$ be a countably closed partially ordered set. Let $\dot{\sigma}$ be a \mathbb{P} -name for a strategy of ONE in the game $G_1^\omega(\Omega_x, \Omega_x)$, played in the generic extension, but on the ground model space X . Thus,

$$\mathbf{1}_{\mathbb{P}} \Vdash \text{“}\dot{\sigma} \text{ is a strategy of ONE in } G_1^\omega(\Omega_{\check{x}}, \Omega_{\check{x}}) \text{ played on } \check{X}\text{”}$$

We must show $\mathbf{1}_{\mathbb{P}} \Vdash \text{“}\dot{\sigma} \text{ is not a winning strategy for ONE.”}$ Thus, let $p \in \mathbb{P}$ be given. We will find a $q < p$ such that q forces that $\dot{\sigma}$ is not a winning strategy for ONE.

By Theorem 1 we have $\mathbf{1}_{\mathbb{P}} \Vdash \text{“}\check{X} \text{ is countably tight.”}$ Thus, we may assume that

$$\mathbf{1}_{\mathbb{P}} \Vdash \text{“for each finite sequence } (\check{x}_1, \dots, \check{x}_n) \text{ from } \check{X}, \dot{\sigma}(\check{x}_1, \dots, \check{x}_n) \text{ is countable”}$$

Define $F(\emptyset) = \{y \in X : (\exists q \leq p)(q \Vdash \text{“}\check{y} \in \dot{\sigma}(\emptyset)\text{”})\}$. As in Claim 1 of Theorem 1 we have in the ground model that $F(\emptyset)$ is an element of Ω_x , and we may assume that $F(\emptyset)$ is countable. Enumerate $F(\emptyset)$ as $(y_n : n < \omega)$ and choose for each n a $q_n < p$ such that $q_n \Vdash \text{“}\check{y}_n \in \dot{\sigma}(\emptyset)\text{”}$.

Then, for each n_1 , define $F(y_{n_1}) = \{y \in X : (\exists q \leq q_{n_1})(q \Vdash \text{“}\check{y} \in \dot{\sigma}(\check{y}_{n_1})\text{”})\}$. As before the ground model set $F(y_{n_1})$ is an element of Ω_x and may be assumed countable, and thus may be enumerated as $(y_{n_1, n} : n < \omega)$. Choose for each n a $q_{n_1, n} < q_{n_1}$ such that $q_{n_1, n} \Vdash \text{“}\check{y}_{n_1, n} \in \dot{\sigma}(\check{y}_{n_1})\text{”}$.

Next, for each (n_1, n_2) define

$$F(y_{n_1}, y_{n_1, n_2}) = \{y \in X : (\exists q \leq q_{n_1, n_2})(q \Vdash \text{“}\check{y} \in \dot{\sigma}(\check{y}_{n_1}, \check{y}_{n_1, n_2})\text{”})\}.$$

Then the ground model set $F(y_{n_1}, y_{n_1, n_2})$ is an element of Ω_x which may be assumed countable. Enumerate $F(y_{n_1}, y_{n_1, n_2})$ as $(y_{n_1, n_2, n} : n < \omega)$. Choose for each n a $q_{n_1, n_2, n} < q_{n_1, n_2}$ such that $q_{n_1, n_2, n} \Vdash \text{“}\check{y}_{n_1, n_2, n} \in \check{\sigma}(\check{y}_{n_1} \check{y}_{n_1, n_2})\text{”}$, and so on.

In this way we define a strategy F for ONE in the ground model. But in the ground model ONE has no winning strategy in $G_1^\omega(\Omega_x, \Omega_x)$. Thus, fix and F -play lost by ONE. This specifies an $f \in {}^\omega\omega$ such that for each n we have $y_{f \upharpoonright n+1} \in F(y_{f(0)}, \dots, y_{f \upharpoonright n})$, and $\{y_{f \upharpoonright n} : n < \omega\} \in \Omega_x$. But since \mathbb{P} is countably closed, choose a $q \in \mathbb{P}$ such that for each n we have $q < q_{f \upharpoonright n}$, $n < \omega$. Then we have $q < p$ and q forces “ $(\check{\sigma}(\emptyset), \check{y}_{f(0)}, \check{\sigma}(\check{y}_{f(0)}), \check{y}_{f(0), f(1)}, \dots, \check{\sigma}(\check{y}_{f(0)}, \dots, \check{y}_{f \upharpoonright n}), \check{y}_{f \upharpoonright n+1}, \dots)$ is a $\check{\sigma}$ -play lost by ONE.” In particular,

$$q \Vdash \text{“}\check{\sigma} \text{ is not a winning strategy for ONE in } G_1(\Omega_{\check{x}}, \Omega_{\check{x}})\text{”}$$

This completes the proof. \square

As a result we obtain the following strengthening of [5] Lemma 5.6:

Corollary 5. *Let (X, τ) be a space which is countably tight at $x \in X$. Let κ be an uncountable cardinal and let $\dot{\mathbb{P}}$ be a $\mathbb{C}(\kappa)$ name for a countably closed poset. Then*

$$1_{\mathbb{C}(\kappa) * \dot{\mathbb{P}}} \Vdash \text{“}\check{X} \text{ has countable strong fan tightness at } \check{x}\text{”}.$$

3. TIGHTNESS AND HFD'S

[6] contains a nice survey of HFD spaces. Note that HFD spaces are subspaces of ${}^\lambda 2$ for appropriate uncountable cardinals λ . As noted in 2.7 of [6], all HFD's are hereditarily separable. The following Lemma must be well-known:

Lemma 6. *If X is a hereditarily separable space, then X has countable tightness at each of its elements.*

Proof. Let $x \in X$ as well as $A \subseteq X$ be given with $x \in \overline{A}$. Since X is hereditarily separable, choose a countable set $B \subseteq A$ which is dense in A . Then $x \in \overline{B}$. \square

Thus, all HFD's are countably tight. In [3] Berner and Juhasz introduce a game denoted $G_\omega^{ND}(X)$, which is played as follows on the topological space X : Players ONE and TWO play an inning per finite ordinal. In inning $n < \omega$ ONE selects a nonempty open subset O_n of X , and TWO responds with a $t_n \in O_n$. A play

$$O_0, t_0, \dots, O_n, t_n, \dots$$

is won by player ONE if the set $\{t_n : n < \omega\}$ is not discrete in X . Else, TWO wins. Let \mathfrak{ND} denote the set $\{A \subseteq X : A \text{ not discrete}\}$, and let \mathfrak{D} denote the set of all dense subsets of X . Using the techniques in Lemma 2 and Theorems 7 and 8 of [12] one can show the following:

- ONE has a winning strategy in $G_\omega^{ND}(X)$ if, and only if, TWO has a winning strategy in $G_1^\omega(\mathfrak{D}, \mathfrak{ND})$.
- TWO has a winning strategy in $G_\omega^{ND}(X)$ if, and only if, ONE has a winning strategy in $G_1^\omega(\mathfrak{D}, \mathfrak{ND})$.

Lemma 7. *Let X be a T_1 -space with no isolated points and assume that ONE has a winning strategy in the game $G_1^\omega(\mathfrak{D}, \mathfrak{ND})$. Then for each $x \in X$ ONE has a winning strategy in the game $G_1^\omega(\Omega_x, \Omega_x)$.*

Proof. Let σ be ONE's winning strategy in $G_1^\omega(\mathfrak{D}, \mathfrak{ND})$. Fix an $x \in X$. Define a strategy σ_x for ONE as follows:

$\sigma_x(\emptyset) = \sigma(\emptyset) \setminus \{x\}$, a dense subset of X since X has no isolated points. With

(t_1, \dots, t_n) a finite sequence of points from X , define $\sigma_x(t_1, \dots, t_n) = \sigma(t_1, \dots, t_n) \setminus \{x\}$ whenever the latter is defined, and else define this to be $X \setminus \{x\}$.

We claim that σ_x is a winning strategy in $G_1^\omega(\Omega_x, \Omega_x)$ for ONE. For consider any σ_x -play

$$O_0, t_0, \dots, O_n, t_n, \dots$$

Then

$$O_0 \cup \{x\}, t_0, \dots, O_n \cup \{x\}, t_n, \dots$$

is a σ -play of $G_1^\omega(\mathfrak{D}, \mathfrak{N}\mathfrak{D})$, and thus the set $S := \{t_n : n < \omega\}$ is discrete. Since $x \notin S$, it follows that S is not a member of Ω_x , and so ONE won the play. \square

Thus we have

Theorem 8 (CH). *There is T_3 space X which has countable strong fan tightness and yet ONE has a winning strategy in the game $G_1^\omega(\Omega_x, \Omega_x)$ at each $x \in X$.*

Proof. In Theorem 3.1 of [3] Berner and Juhasz construct, using CH, an HFD $\mathbb{J} \subseteq {}^{\omega_1}2$ with no isolated points such that ONE has a winning strategy in $G_1^\omega(\mathfrak{D}, \mathfrak{N}\mathfrak{D})$. The following Theorem 10 implies that each HFD has countable strong fan tightness. Apply this information and Lemma 7 to the HFD \mathbb{J} . \square

At first glance Theorem 8 might suggest that there is an HFD with destructible countable tightness. We now modify Theorem 2.7 of [3] to show that this is in fact not the case. For the convenience of the reader we recall:

If $X \subseteq {}^\lambda 2$ is an HFD define for S a countably infinite subset of X the set D_S to be the set of all countable $A \subseteq \lambda$ such that for all $\sigma \in \text{Fin}(A, 2)$, if $[\sigma] \cap S$ is infinite then for each $\tau \in \text{Fin}(\lambda \setminus A, 2)$ the set $[\sigma] \cap [\tau] \cap S$ is nonempty. It is known (see [6], 2.12) that:

Lemma 9. *For each countably infinite subset S of an HFD $X \subseteq {}^\lambda 2$ the set $D_S \subseteq [\lambda]^{\aleph_0}$ is closed and unbounded.*

Theorem 10. *If $X \subseteq {}^\lambda 2$ is an HFD then for each $x \in X$ TWO has a winning strategy in $G_1^{\omega^2}(\Omega_x, \Omega_x)$.*

Proof. Fix an $x \in X$. We may assume that x is not an isolated point of X .

Choose $B_1 \subset \lambda$ countably infinite with $\omega \subseteq B_1$ and $\{\sigma \in \text{Fin}(B_1, 2) : x \in [\sigma]\}$ infinite. Let $(\sigma_n^1 : 0 < n < \omega)$ enumerate $\{\sigma \in \text{Fin}(B_1, 2) : x \in [\sigma]\}$ in such a way that each element is listed infinitely many times.

We now describe a strategy Φ for player TWO: During the first ω innings, when ONE plays in inning i a set $O_i \in \Omega_x$, TWO plays

$$\Phi(O_1, \dots, O_i) \in [\sigma_i^1] \cap O_i \setminus \{\Phi(O_1, \dots, O_j) : j < i\}.$$

Put $A_1 = \{\Phi(O_1, \dots, O_k) : 0 < k < \omega\}$. Then A_1 is an infinite subset of X . If $A_1 \in \Omega_x$, then TWO plays the remaining innings arbitrarily, obeying the rules of the game. Thus, assume that $A_1 \notin \Omega_x$. Towards defining Φ for the next ω innings of the game, choose $C_1 \in D_{A_1}$ with B_1 a proper subset of C_1 and let $(\sigma_n^2 : 0 < n < \omega)$ enumerate $\{\sigma \in \text{Fin}(C_1, 2) : x \in [\sigma]\}$ such that each element is listed infinitely often. Now when ONE plays $O_{\omega+i}$, $i < \omega$ TWO plays

$$\Phi(O_\nu : \nu \leq \omega + i) \in [\sigma_i^2] \cap O_{\omega+i} \setminus \{\Phi(O_\gamma : \gamma < j) : j \leq \omega + i\}$$

which is possible as $A_1 \notin \Omega_x$ and x is not an isolated point of X .

Put $A_2 = A_1 \cup \{\Phi(O_\gamma : \gamma < j) : j < \omega \cdot 2\}$. If A_2 is a member of Ω_x , then TWO plays arbitrary points during the rest of the game, following the rules of the game.

Thus, assume that A_2 is not a member of Ω_x . Choose $C_2 \in D_{A_1} \cap D_{A_2}$ with C_1 a proper subset of C_2 and let $(\sigma_n^3 : 0 < n < \omega)$ enumerate $\{\sigma \in \text{Fin}(C_2, 2) : x \in [\sigma]\}$ such that each element is listed infinitely often. Now when ONE plays $O_{\omega \cdot 2 + i}$, $i < \omega$ TWO plays

$$\Phi(O_\nu : \nu \leq \omega \cdot 2 + i) \in [\sigma_i^2] \cap O_{\omega+i} \setminus \{\Phi(O_\gamma : \gamma < j) : j \leq \omega \cdot 2 + i\}$$

which is possible as $A_2 \notin \Omega_x$ and x is not an isolated point.

Then put $A_3 = A_2 \cup \{\Phi(O_\nu : \nu \leq \omega \cdot 2 + i) : i < \omega\}$. Choose $C_3 \in D_{A_1} \cap D_{A_2} \cap D_{A_3}$ with C_2 a proper subset of C_3 , and so on. TWO continues playing like this until an n is reached at which $A_n \in \Omega_x$, and then plays arbitrary points permitted by the rules of the game.

Suppose that for each n we have $A_n \notin \Omega_x$. Consider $A = \bigcup_{0 < k < \omega} A_k$. The set A came about through TWO using the strategy Φ above over ω^2 innings.

Claim: $x \in \overline{A}$.

To see that A meets each neighborhood of x , let a $\sigma \in \text{Fin}(\lambda, 2)$ be given with $x \in [\sigma]$. Also, put $C = \bigcup \{C_n : 0 < n < \omega\}$. Since each D_{A_j} is closed and unbounded, C is an infinite member of $\bigcap_{0 < n < \omega} D_{A_n}$. Since the containments $C_n \subset C_{n+1}$ are proper and since $\text{dom}(\sigma)$ is finite, choose the least $k < \omega$ with $\text{dom}(\sigma) \cap C_k = \text{dom}(\sigma) \cap C$. Put $\nu = \sigma \upharpoonright_{C_k}$. Now for each $m \geq k$, ν was listed infinitely often among $\{\sigma \in \text{Fin}(C_m, 2) : x \in [\sigma]\}$, implying that $[\nu] \cap A_m$ is infinite and in particular nonempty. This implies that $[\sigma] \cap A \neq \emptyset$. It follows that Φ is a winning strategy for TWO. \square

Corollary 11. *Each HFD is indestructibly countably tight.*

Proof. Apply Theorems 1 and 10. \square

It was shown in Proposition 19 of [12] that HFD's satisfy $S_1(\mathfrak{D}, \mathfrak{D})$. We shall now see that these prior results follow from Theorem 10. First recall that for families \mathcal{A} and \mathcal{B} of sets, $S_{fin}(\mathcal{A}, \mathcal{B})$ denotes that there is for each sequence $(A_n : n < \omega)$ of elements of \mathcal{A} a sequence $(B_n : n < \omega)$ where for each n , $B_n \subseteq A_n$ is a finite set, and $\bigcup \{B_n : n < \omega\}$ is an element of \mathcal{B} . The notion of *countable fan tightness* was introduced by Arkhangel'skii in [1]: A space X is countably fan tight at $x \in X$ if the selection principle $S_{fin}(\Omega_x, \Omega_x)$ holds at x .

Lemma 12. *If X has countable (strong) fan tightness at each of its elements, and if X is separable, then X has the properties $S_{fin}(\mathfrak{D}, \mathfrak{D})$ (respectively, $S_1(\mathfrak{D}, \mathfrak{D})$).*

Proof. Let D be a fixed countable dense subset of X , enumerated bijectively as $(d_n : n < \omega)$. Let $(D_n : n < \omega)$ be a sequence of dense subsets of X . Write $\omega = \bigcup_{m \in \omega} S_m$ where each S_m is infinite and for $m < n$, $S_m \cap S_n = \emptyset$. Now for each m we have $x_m \in \overline{D_n}$, $n \in S_m$. If X has countable strong fan tightness at each of its elements, choose for $n \in S_m$ an $x_n \in D_n$ such that $x \in \overline{\{x_n : n \in S_m\}}$, using countable strong fan tightness at each point. But then $\{x_n : n \in \omega\}$ is dense, witnessing $S_1(\mathfrak{D}, \mathfrak{D})$. The argument for countable fan tightness is similar. \square

Corollary 13. *If X is a separable space which is countably tight at each element and if κ is an uncountable cardinal, then*

$$1_{\mathbb{C}(\kappa)} \Vdash \text{“}\tilde{X} \text{ satisfies } S_1(\mathfrak{D}, \mathfrak{D})\text{”}.$$

Corollary 14. *If X is a hereditarily separable space and κ is an uncountable cardinal, then*

$$1_{\mathbb{C}(\kappa)} \Vdash \text{“}\tilde{X} \text{ satisfies } S_1(\mathfrak{D}, \mathfrak{D})\text{”}.$$

By the technique used in the proof of Theorem 3 each of these corollaries can be strengthened to:

Proposition 15. *If X is a separable space which is countably tight at each element and if κ is an uncountable cardinal, then*

$$1_{\mathbb{C}(\kappa)} \Vdash \text{“ONE has no winning strategy in } G_1^\omega(\mathfrak{D}, \mathfrak{D}) \text{ on } \check{X}.”}$$

4. HOMOGENEOUS T_5 COMPACTA.

In Theorem 2.8 of [7] it is proved that in the generic extension obtained by forcing with \aleph_2 Cohen reals, every homogeneous compact T_5 space is countably tight and of character at most \aleph_1 . We now show:

Theorem 16. *In the generic extension obtained by forcing with $\mathbb{C}(\aleph_2)$, for each homogeneous compact T_5 -space X , ONE has no winning strategy in $G_1^\omega(\Omega_x, \Omega_x)$ at each $x \in X$.*

Proof. The proof uses the idea appearing in the proof of [7], Theorem 2.8, that “first adding \aleph_2 Cohen reals and then adding \aleph_1 is equivalent to adding \aleph_2 Cohen reals all at once.” Consider a homogeneous T_5 -compactum in the generic extension by \aleph_2 Cohen reals. It has countable tightness. Now add another \aleph_1 Cohen reals, and apply Theorem 3. \square

Thus, in the Cohen model all homogeneous T_5 compacta have countable strong fan tightness and are indestructibly countably tight.

5. GENERIC LEFT-SEPARATED SPACES.

Let ν be an ordinal. In Section 2 of [8] it is proved that in the generic extension $V^{\mathbb{P}_\nu}$ by a special complete suborder \mathbb{P}_ν of the Cohen partially ordered set $\text{Fn}(\nu \times \nu, 2)$, there is a topology τ on ν such that the space $X_\nu = (\nu, \tau)$ has the following properties:

- (a) X_ν is hereditarily Lindelöf (Lemma 2.1);
- (b) X_ν has countable tightness (Lemma 2.2);
- (c) The density of X_ν is $\text{cof}(\nu)$ (Lemma 2.3);
- (d) X_ν is left-separated in the natural well-ordering of ν .

We will now argue that if ν is an uncountable regular cardinal, then X_ν is (1) an L-space which is hereditarily a Rothberger space (and thus indestructibly Lindelöf and D-space), and (2) ONE has no winning strategy in the game $G_1^\omega(\Omega_x, \Omega_x)$ at each $x \in X_\nu$, so that X_ν is indestructibly countably tight and has countable strong fan tightness.

The fact that X_ν is an L-space when ν is a regular uncountable cardinal was noted in [8]. Regarding the rest of the claimed properties of X_ν :

The first point is that if ν is a regular uncountable cardinal then the partial order \mathbb{P}_ν densely embed into $\text{Fn}(\kappa, 2)$, so that the generic extension is the same as the generic extension by $\mathbb{C}(\kappa)$. Thus X_κ is obtained by adding κ Cohen reals. To see this we review the definition of \mathbb{P}_κ . An element p of $\text{Fn}(\kappa \times \kappa, 2)$ is an element of \mathbb{P}_κ if, and only if, it has the following two properties:

- (1) If (α, α) is in the domain of p , then $p(\alpha, \alpha) = 1$;
- (2) If (α, β) is in the domain of p and $p(\alpha, \beta) = 1$, then $\alpha \leq \beta$.

Now let S be the subset of $\kappa \times \kappa$ consisting of the pairs (α, β) with $\alpha < \beta$. Then the mapping

$$F : P_\kappa \longrightarrow \text{Fn}(S, 2)$$

defined by $F(p) = p \restriction_S$, the restriction of p to S , is a dense embedding (See [9], Definition VII.7.7). Thus, as $|S| = \kappa$, the generic extension by \mathbb{P}_κ is the same as the generic extension by $\mathbb{C}(\kappa)$.

The second point is that since κ is regular and uncountable, first adding κ Cohen reals and then \aleph_1 Cohen reals is equivalent to adding κ Cohen reals at once. Now apply Theorem 3 to conclude that ONE has no winning strategy in $\mathbf{G}_1^\omega(\Omega_x, \Omega_x)$ at each x in X_κ , and then Theorem 1 to conclude that X_κ is indestructibly countably tight. Also apply Theorem 11 of [13] to conclude that X_κ is Rothberger, and Corollary 10 of [13] to conclude that X_κ is indestructibly Lindelöf. By results of Aurichi [2] the fact that X_κ is Rothberger implies that X_κ is a D-space.

6. EXAMPLE: DESTRUCTIBLY COUNTABLY TIGHT SPACES

We give an example of the destruction of countable tightness by countably closed forcing. It is well known that ${}^{\omega_1}2$ is destructibly Lindelöf. Since it is compact in all finite powers, it is Lindelöf in all finite powers and so by the Arkhangel'skii-Pytkeev Theorem, $\mathbf{C}_p({}^{\omega_1}2)$ has countable tightness. In fact, by a result of Arkhangel'skii, this space has countable fan tightness.

We shall now see that the countable tightness of this space is destructible by countably closed forcing: We force with $\mathbb{P} = \text{Fn}(\omega_1, 2, \omega_1)$. In the ground model define for each $\alpha < \omega_1$ and $i \in \{0, 1\}$ the open set $U_i^\alpha = \{f \in {}^{\omega_1}2 : f(\alpha) = i\}$. Let $({}^{\omega_1}2)_G$ denote the ground model version of ${}^{\omega_1}2$. If f is \mathbb{P} -generic, then

$$\mathcal{U} = \{U_{f(\alpha)}^\alpha : \alpha < \omega_1\}$$

is an open cover of $({}^{\omega_1}2)_G$, and has no countable subset that covers $({}^{\omega_1}2)_G$. Let \mathcal{V} be the set of finite unions of elements of \mathcal{U} . Then \mathcal{V} is an open ω -cover of $({}^{\omega_1}2)_G$, but it has no countable subset which covers $({}^{\omega_1}2)_G$. Now each $V \in \mathcal{V}$ is a ground model open set and thus for each $x \in ({}^{\omega_1}2)_G \setminus \overline{V}$ there is a ground model continuous function f_U such that $f_U[\overline{U}] \subseteq \{0\}$ and $f_U(x) = 1$. The set

$$\{f_U : U \in \mathcal{V}\}$$

(in the generic extension) is a subset of the ground model version of the set $\mathbf{C}_p({}^{\omega_1}2)$. In the generic extension this uncountable set has the zero function in its closure, but no countable subset of it does since f is generic. Thus the ground model version of the set $\mathbf{C}_p({}^{\omega_1}2)$, which had countable fan tightness in the ground model, is not countably tight in the extension.

Not all is lost when countable tightness is destroyed by a countably closed partial order: A topological space (X, τ) is said to have *countable extent* if each closed, discrete subspace of X is countable. The Lindelöf property implies countable extent. Tall ([15], Lemma 8), and independently Dow [4], proved:

Lemma 17 (Dow, Tall). *If a topological space has countable tightness, then in generic extensions by countably closed partially ordered sets the space has countable extent.*

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